ZetaGrid

Computations Connected with the Verification of the Riemann Hypothesis

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The Riemann Hypothesis

- **Let s be a complex number with Re(s)>1. Then the Riemann zeta function is defined by**

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

and is extended to the rest of the complex plane (except for s=1) by analytic continuation.

- **The Riemann Hypothesis (formulated in 1859) asserts that all nontrivial zeros of the zeta function are on the critical line (1/2+*it* where *t* is a real number).**

- **The last 140 years did not bring its proof or disproof. In 2000, Clay Mathematics Institute offered a $1 million prize for proof of the Riemann Hypothesis.**

Proved Facts about Riemann's Hypothesis

- Infinitely many zeros lie on the critical line.
- At least 40% of all nontrivial zeros lie on the critical line.
- All first 75 billion zeros are simple and lie on the critical line, thus, the Riemann Hypothesis is true at least for all $|\mathrm{Im}(s)| < 22{,}455{,}199{,}960.64$.

History of Computed Zeros (Part I)

Year	Author	Number of zeros
1903	J. P. Gram	15
1914	R. J. Backlund	79
1925	J. I. Hutchinson	138
1935	E. C. Titchmarsh	1,041
1953	A. M. Turing	1,104
1955	D. H. Lehmer	10,000
1956	D. H. Lehmer	25,000
1958	N. A. Meller	35,337
1966	R. S. Lehman	250,000
1968	J. B. Rosser, J. M. Yohe, L. Schoenfeld	3,500,000
1977	R. P. Brent	40,000,000
1979	R. P. Brent	81,000,001
1982	R. P. Brent, J. van de Lune, H. J. J. te Riele, D. T. Winter	200,000,001
1983	J. van de Lune, H. J. J. te Riele	300,000,001
1986	J. van de Lune, H. J. J. te Riele, D. T. Winter	1,500,000,001
2001	J. van de Lune	10,000,000,000
2002	S. Wedeniwski	50,631,912,399

History of Computed Zeros (Part II)

- In 1988, a faster method for simultaneous computation of large sets of zeros of the zeta function was invented by A. M. Odlyzko and A. Schönhage.
- It has been implemented and used to compute 175×10^6 zeros near zero number 10^{20} (1992),
- 10 billion zeros near zero number 10^{22} (2001),
- and about 20 billion zeros near zero number 10^{23} (2002).

The Function *Z(t)*

- **Let**

$$Z(t) = e^{i\theta(t)}\zeta(\tfrac{1}{2} + it)$$

and

$$\theta(t) = \mathrm{Im}(\ln(\Gamma(\tfrac{1+2it}{4}))) - \tfrac{t}{2}\ln(\pi).$$

- **Then the function *Z(t)* is real for real *t*, and**

$$|Z(t)| = |\zeta(\tfrac{1}{2} + it)|.$$

The Riemann-Siegel Formula (Part I)

- **Let *t* be a real number,**

$$\tau = \frac{t}{2\pi}, m = \lfloor\sqrt{\tau}\rfloor, \text{and } z = 2(\sqrt{\tau}-m)-1.$$

- **Then the Riemann-Siegel formula with four terms in the asymptotic expansion is**

$$\begin{aligned} Z(t) &= 2\sum_{k=1}^{m}\frac{1}{\sqrt{k}}\cdot\cos(t\ln(k)-\theta(t)) \\ &\quad +(-1)^{m+1}\sum_{j=0}^{3}\Phi_j(z)(-1)^j\tau^{-\frac{2j+1}{4}} \\ &\quad +O(\tau^{-9/4}). \end{aligned}$$

The Riemann-Siegel Formula (Part II)

- Here the $F_j(z)$ are certain entire functions which may be expressed by

$$\Phi_0(z) = \frac{\cos\left(\frac{\pi(4z^2+3)}{8}\right)}{\cos(\pi z)}$$

$$\Phi_1(z) = \frac{\Phi_0^{(3)}(z)}{12\pi^2}$$

$$\Phi_2(z) = \frac{\Phi_0^{(2)}(z)}{16\pi^2} + \frac{\Phi_0^{(6)}(z)}{288\pi^4}$$

$$\Phi_3(z) = \frac{\Phi_0^{(1)}(z)}{32\pi^2} + \frac{\Phi_0^{(5)}(z)}{120\pi^4} + \frac{\Phi_0^{(9)}(z)}{10368\pi^6}.$$

- $F_j(z)$ can be expressed by the Taylor series expansions.

Gram Points

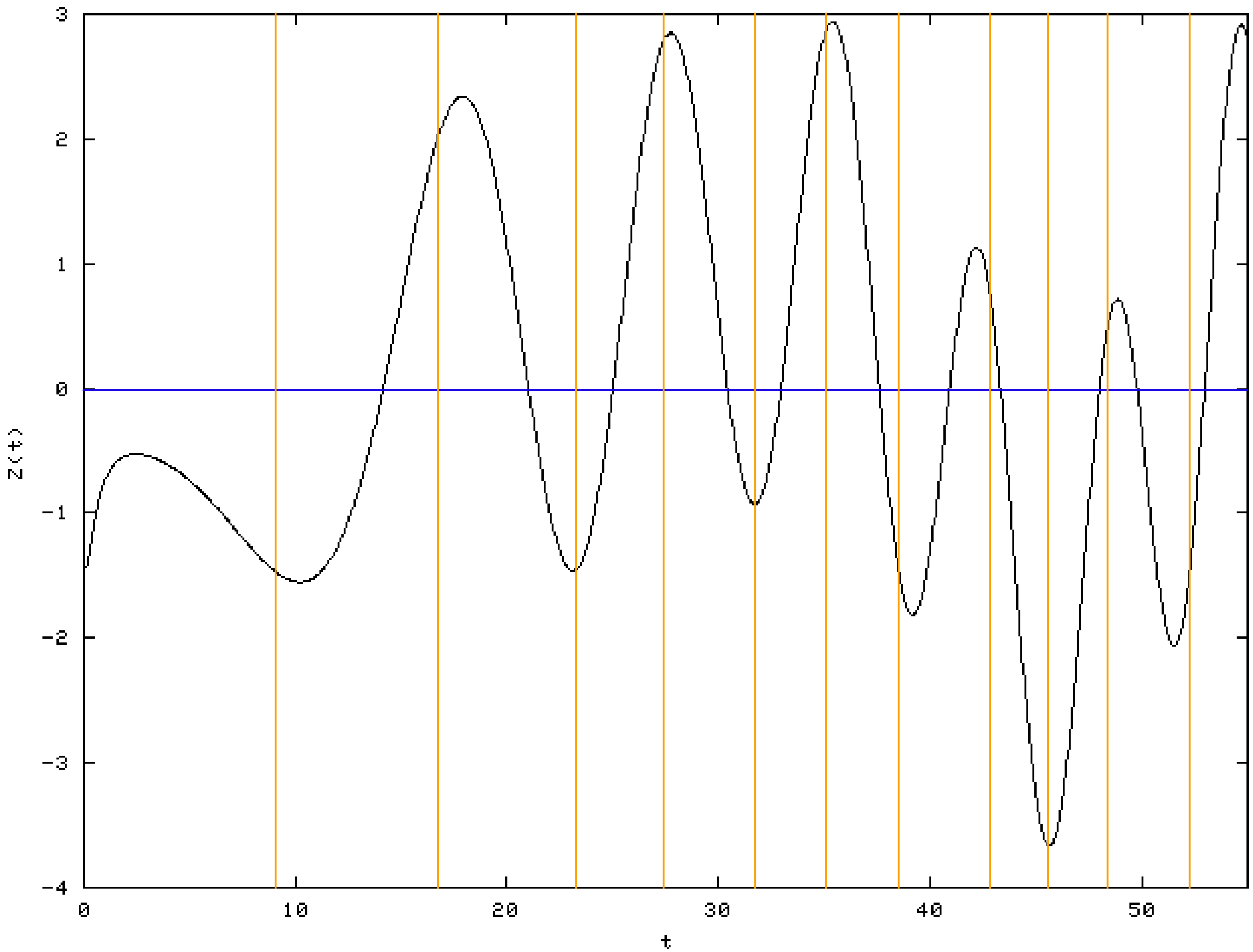
- **Let $m \geq -1$ be an integer. The mth Gram point g_m is defined as a unique solution of the equation**

$$\theta(g_m) = m\pi.$$

- **Thus, e.g.,**

$$\begin{aligned} g_{-1} &\in \,]9.67690678, 9.67690679[\\ g_0 &\in \,]17.84783651, 17.84783652[\\ g_1 &\in \,]23.17166081, 23.17166082[\\ g_2 &\in \,]27.67119803, 27.67119804[\end{aligned}$$

The Zeros of *Z(t)*



The first 11 non-trivial zeros:

r_1 » ***1/2 + 14.135i***
r_2 » ***1/2 + 21.022i***
r_3 » ***1/2 + 25.011i***
r_4 » ***1/2 + 30.425i***
r_5 » ***1/2 + 32.935i***
r_6 » ***1/2 + 37.586i***
r_7 » ***1/2 + 40.919i***
r_8 » ***1/2 + 43.327i***
r_9 » ***1/2 + 48.005i***
r_{10} » ***1/2 + 49.774i***
r_{11} » ***1/2 + 52.970i***

Using Rosser Blocks to Find Sign Changes of *Z(t)*, Part I

- **Let m ^{3} -1 be an integer.**
  Gram's law **is the observation of J. P. Gram (1903) that *Z(t)* usually changes sign in each *Gram interval***

$$G_m = [g_m, g_{m+1}[\ .$$

- **This suggested "law," as J. P. Gram himself suspected, is false although the first failure does not occur until the Gram point g_{125} » 280.80.**

Using Rosser Blocks to Find Sign Changes of *Z(t), Part II*

- **J. B. Rosser, J. M. Yohe and L. Schoenfeld introduced (1968) the concept of *Rosser blocks* as *Rosser's rule* to handle the failures of Gram's law.**
  The first exeption is $B_{13,999,525}$ » 6,820,050.98.

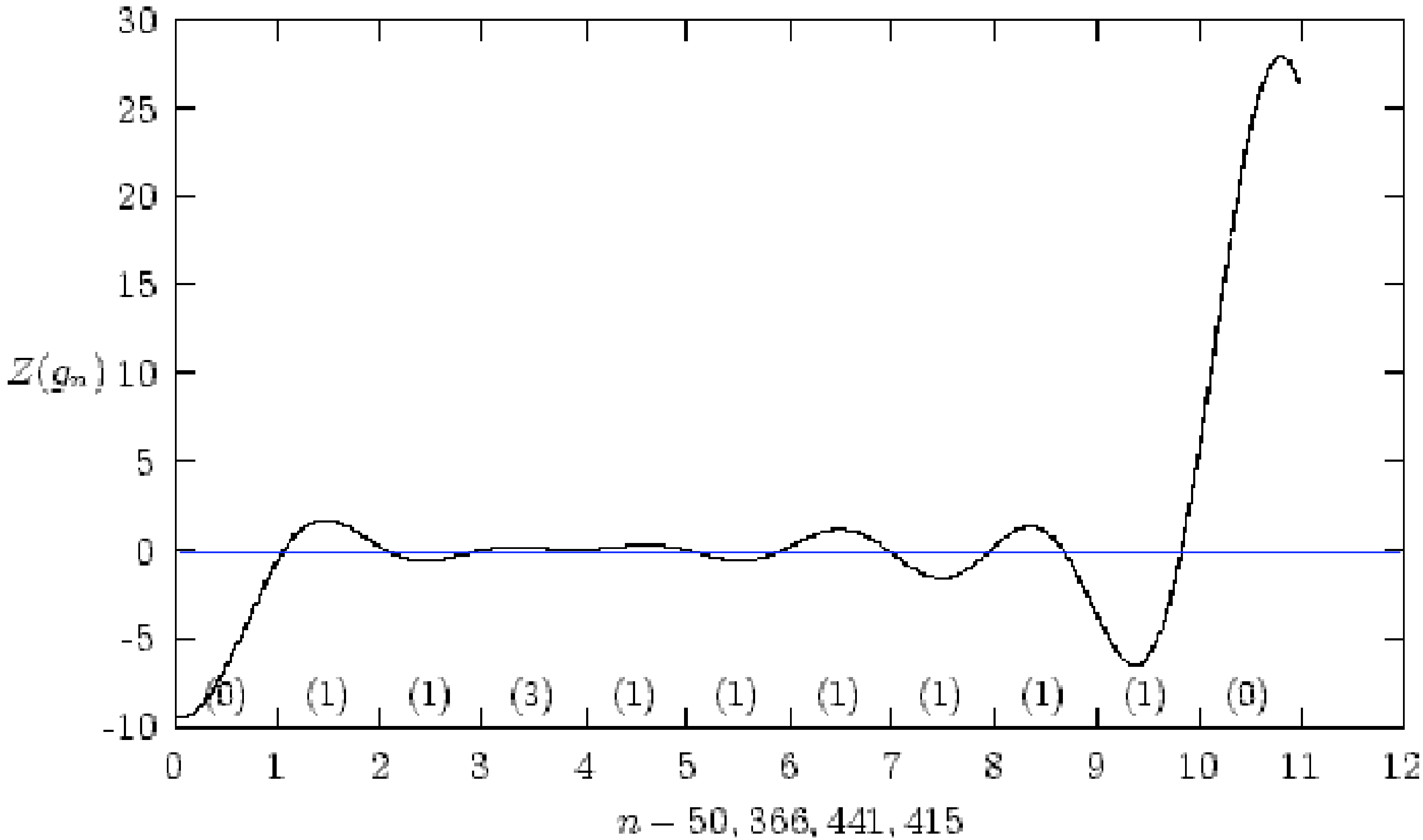
- **A Rosser block of length *k* is an interval**
  B_m=[g_m, g_{m+k}[such that $(-1)^m Z(g_m) > 0$, $(-1)^{m+k} Z(g_{m+k}) > 0$
  and $(-1)^{m+l} Z(g_{m+l}) < 0$ for 1 £ *l* < *k* and *k* ³ 1.

- **We say that B_m satisfies *Rosser's rule* if *Z(t)* has at least *k* zeros in B_m.**

- **Rosser's rule fails infinitely often (R. S. Lehman 1970), but it is still an extremely useful heuristic. Using this rule, we achieved the average number 1.26 of *Z*-evaluations which is needed to separate a zero from its predecessor.**

$B_{50,366,441,415}$ is the First Rosser Block of Length 11

- The Rosser block $B_{50,366,441,415}$ has a zero-pattern '01131111110.'



Determining all Zeros of z(s)

- Let *N*(*T*) denotes the number of nontirvial zeros of z(*s*) in the region 0 < Im(*s*) £ T, and

$$S(T) = N(T) - 1 - \frac{\theta(T)}{\pi}.$$

- Gram's law holds in regions where |*S*(*t*)| < 1, and Rosser's rule holds in regions where |*S*(*t*)| < 2.
  We have $S(g_k)$ » 3.0214 for *k* = 53,365,784,979
  and $S(g_k)$ » -3.2281 for *k* = 67,976,501,145.

- R. P. Brend (1979) showed the following theorem, based on an idea of J. E. Littlewood and a theorem of A. M. Turing:

  If *K* consecutive Rosser blocks with union [g_m, g_n[satisfy Rosser's rule, then

  $N(g_m)$ £ *m*+1 and $N(g_n)$ ³ *n*+1.

Computational Aspects

- About 99.7% of the running time was spent on evaluating the sum

$$\sum_{k=1}^{m} \frac{1}{\sqrt{k}} \cdot \cos(t \ln(k) - \theta(t))$$

- ln(*k*) is looked up in a precomputed table.
  $\theta(t)$ is evaluated one time per sum with twice the precision of IEEE double (106 mantissa bit).

- In our program we used a very fast and efficient method and a comperatively slow (factor 120) but very accurate method. Both methods give the correct sign of *Z*(*t*) if the evaluated lower and upper bound has the same sign.

- In the fast method, the cosine-values are approximated by the table of $\cos(2\pi \cdot k/2^{20})$, $0 \leq k \leq 2^{20}$, and we have the unit roundoff $\epsilon := 2^{-63}$ using accurate IEEE floating-point arithmetic.

Computing Rigorous the Lower and Upper Bound of $Z(t)$

- **Let $l(t, k) := (t \cdot \ln(k) - \theta(t)) \bmod 2\pi$ and let $l'(t, k)$ be the computed value of $l(t, k)$. Than we can show that**

  $|l(t, k) - l'(t, k)| < 5\varepsilon = 5 \cdot 2^{-63}$ **or** $|l(t, k) - l'(t, k) - 2\pi| < 5\varepsilon.$

- **Let $c'(k)$ be the approximated cosine-values for $0 \le k \le 2^{20}$. Then the lower bound of $Z(t)$ is evaluated by**

$$\min\left\{c'\left(\left\lfloor\frac{l'(t,k)}{2^{20}}\right\rfloor - 1\right), c'\left(\left\lfloor\frac{l'(t,k)}{2^{20}}\right\rfloor + 1\right)\right\}$$

  and the higher bound by

$$\max\left\{c'\left(\left\lfloor\frac{l'(t,k)}{2^{20}}\right\rfloor - 1\right), c'\left(\left\lfloor\frac{l'(t,k)}{2^{20}}\right\rfloor + 1\right)\right\}$$

Statistic Concerning Rosser Blocks

- **The following tables gives the number of Rosser blocks of length k £ 4 in the interval [g_0, $g_{75,039,937,803}$[for strings of 10^{10} successive Gram intervals $J(k, n+10^{10})-J(k, n)$:**

$\lfloor (n-1)/10^{10} \rfloor$	$k = 1$	2	3	4
0	6800807304	1061761782	257082304	61342210
1	6727060564	1063103749	265495413	68145485
2	6701634358	1063334116	268191756	70453258
3	6685595060	1063414660	269870001	71887721
4	6673792597	1063484462	271072687	72934993
5	6664616290	1063528524	272010773	73740299
6	6657063763	1063498345	272784334	74414225
Totals	46910569936	7442125638	1876507268	492918191
%	82.53	13.09	3.30	0.87

where $J(k, n)$ gives the number of Rosser blocks B_j of length k with 1 £ $j < n$.

Zero-Patterns of the Rosser Blocks (Part I)

- **Let *M*(*P*) be the set of all Rosser blocks with the zero-pattern *P*. Then the following three tables give the number and the first occurrences of a Rosser block with zero-pattern *P* in the interval [g_0, $g_{75,039,937,803}$[:**

P	$\min(M(P))$	$\lvert M(P)\rvert$
1	−1	50262557666
02	125	3988674865
20	133	3988616830
012	4921	960246633
210	3356	960213358
030	2144	93712289
0112	67433	248483837
2110	83701	248453529
0130	18243	16851637
0310	39889	16848453
01112	455256	50172421
21110	1833652	50168978
01130	68084	4245005
03110	243021	4244224
01310	601944	2382998

Zero-Patterns of the Rosser Blocks (Part II)

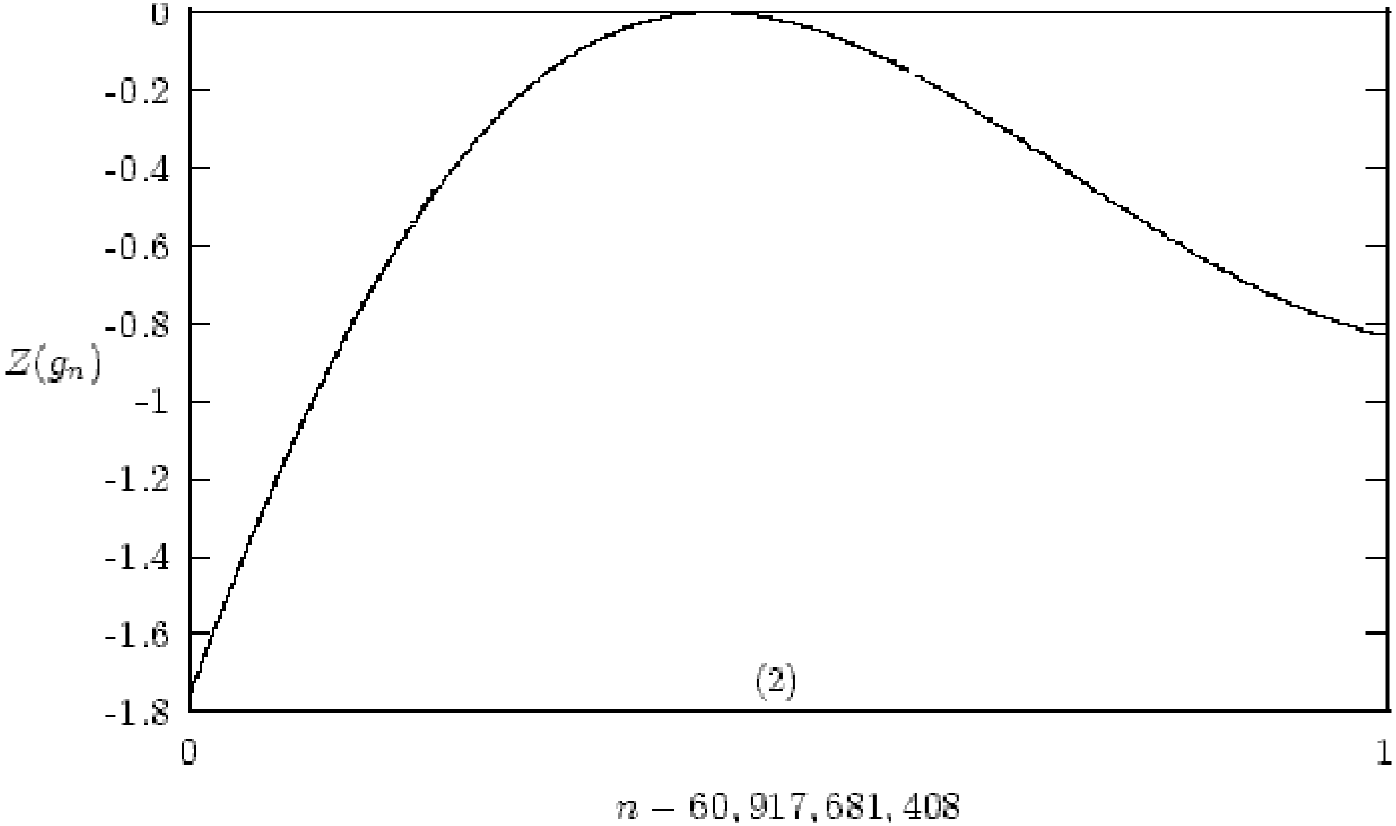
- **1981 J. van de Lune, H. J. J. te Riele and D. T. Winter suggested the following heuristic:**
  Searching the "missing two zeros" is performed in zig-zag manner, moving from the periphery of the block towards its center.

Zero-Patterns of the Rosser Blocks (Part III)

P	$\min(M(P))$	$\lvert M(P)\rvert$
011112	19986469	5804318
211110	20046223	5802000
031110	2656216	1162742
011130	2842089	1161317
013110	4718714	459470
011310	1181229	458172
0111112	195610937	268757
2111110	258779892	267841
0311110	13869654	251757
0111130	52266282	251274
0131110	17121221	98748
0111310	20641464	98570
0113110	37091042	56958
03111110	112154948	26826
01111130	165152519	26782
01311110	175330804	20790
01111310	145659810	20539
01113110	454025825	6671
01131110	717574239	6639
01111112	5987600114	2904
21111110	2894423255	2855

Close Zeros

- $B_{60,917,681,408}$, the Rosser block containing the closest observed pair of zeros



- The distance between these two zeros is less than 0.0001008.

History and Milestones of ZetaGrid

- Feb. 1998 to Nov. 2001: Working on the necessity of the Extended Riemann Hypothesis for Miller's primality test (Dissertation "Primality Tests on Commutator Curves")
- February 2001: First implementation of ZetaGrid and synchronization with the Fortran-Code of J. van de Lune, H. J. J. te Riele, D. T. Winter
- August 2001: Starting ZetaGrid on 10 computers in IBM Laboratory Böblingen
- February 2002: Distributing ZetaGrid on 500 computers in IBM Germany
- September 2002: Availibility of ZetaGrid in the internet at http://www.zetagrid.net

Idea: Use Idle Resources ...

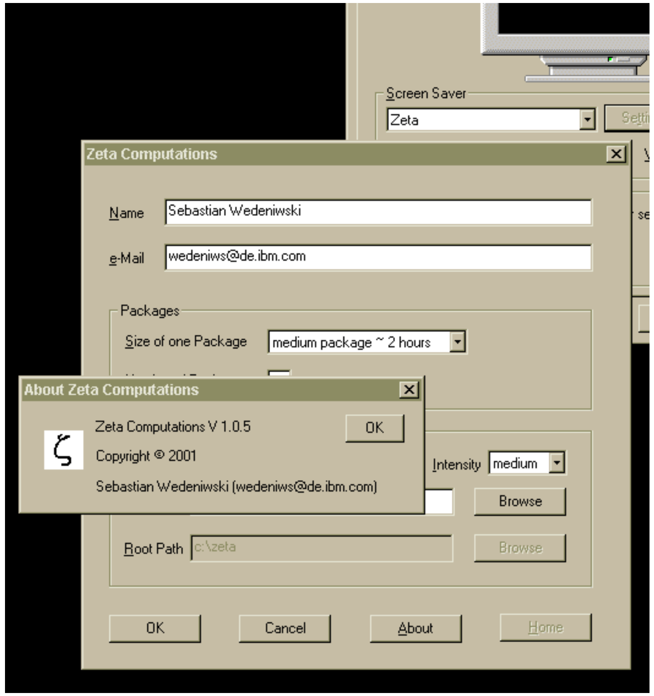
- About 90% of the CPU capability of an office computer is unused.

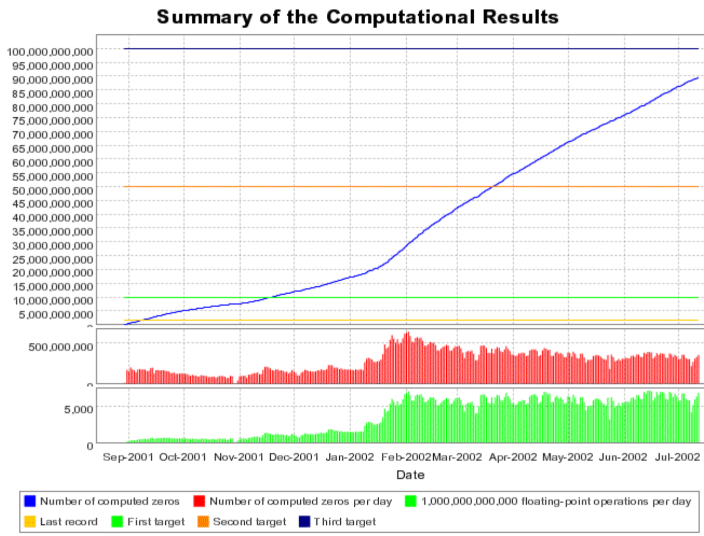
- Room for additional computations
- CPU power is available for free
- Central control, simple administration, good scalability

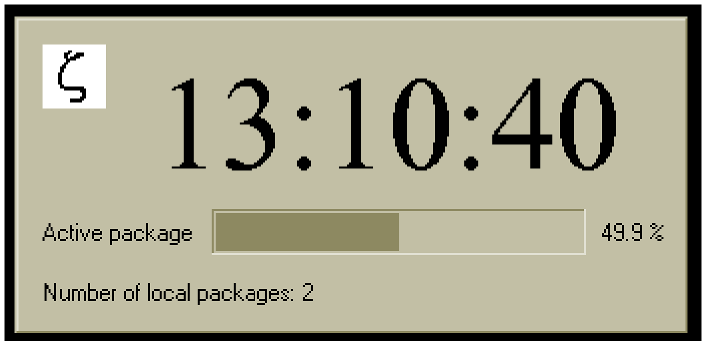
...Over the Internet

ZetaGrid

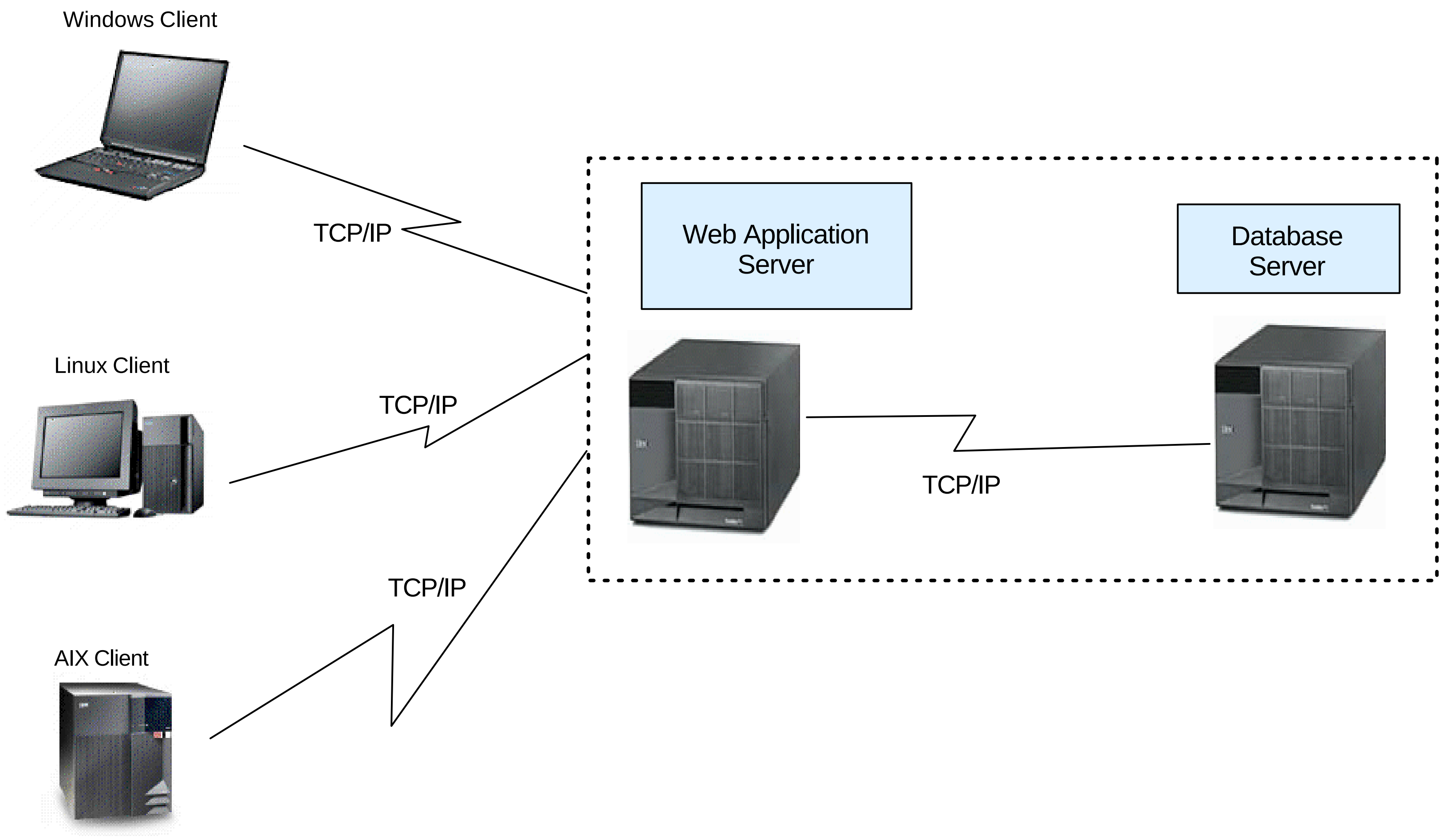
- Runs as screen saver or low-priority process on Windows / AIX / Linux (also z-Series).
- Downloads 'tasks' using just the HTTP protocol and computes them on the local computer.
- Flexible embedding of tasks via API.
- Secured server, back-end database DB2, monitoring, statistics, and full audit trail are available.
- Security protocols and methods for authorization (tasks, results).
- Proven and stable, runs at IBM Lab with about 600 participating computers without any problems to compute mathematical problems (Riemann Hypothesis).



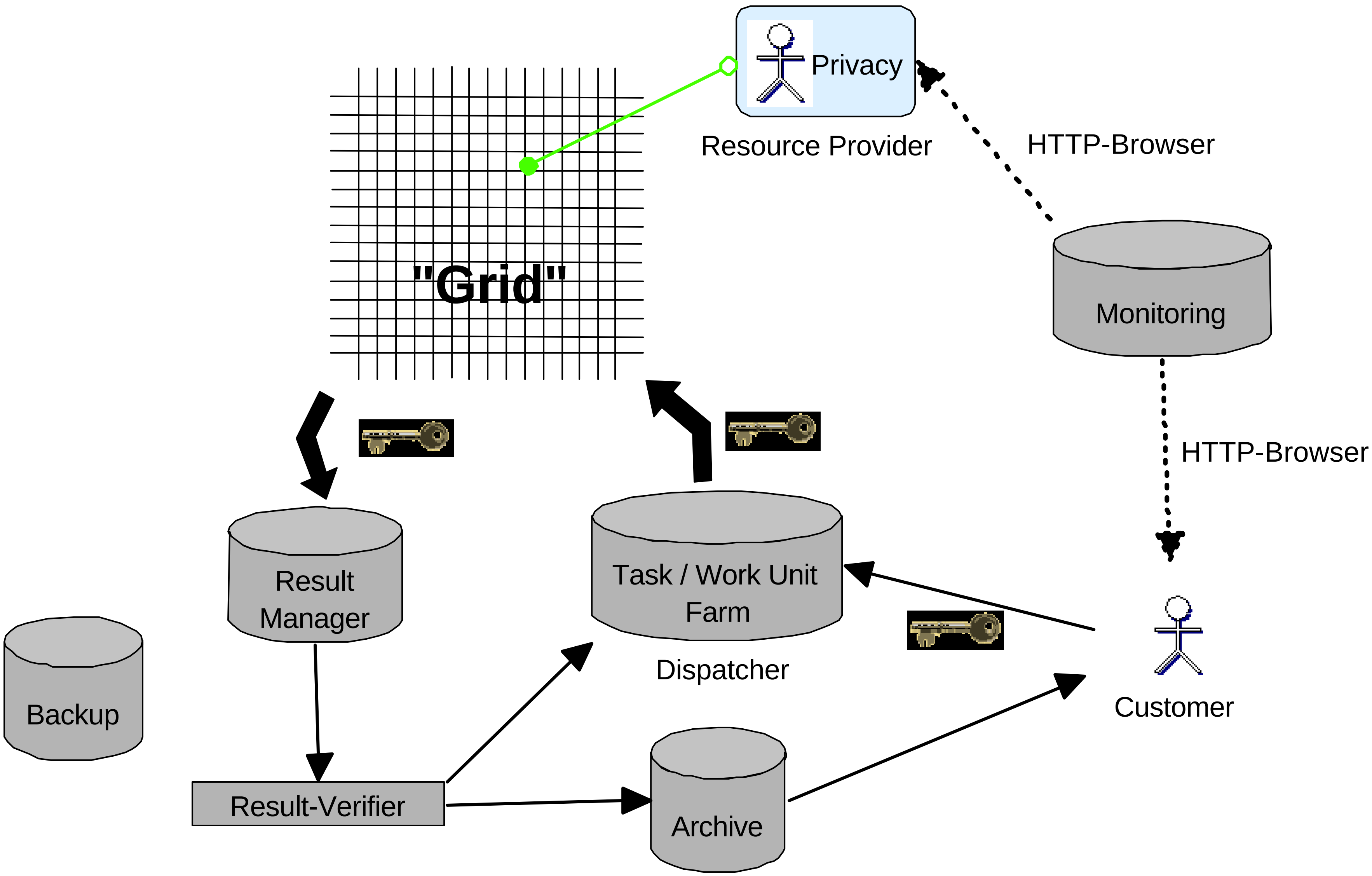




ZetaGrid System Architecture



ZetaGrid Security Architecture



Performance Characteristics

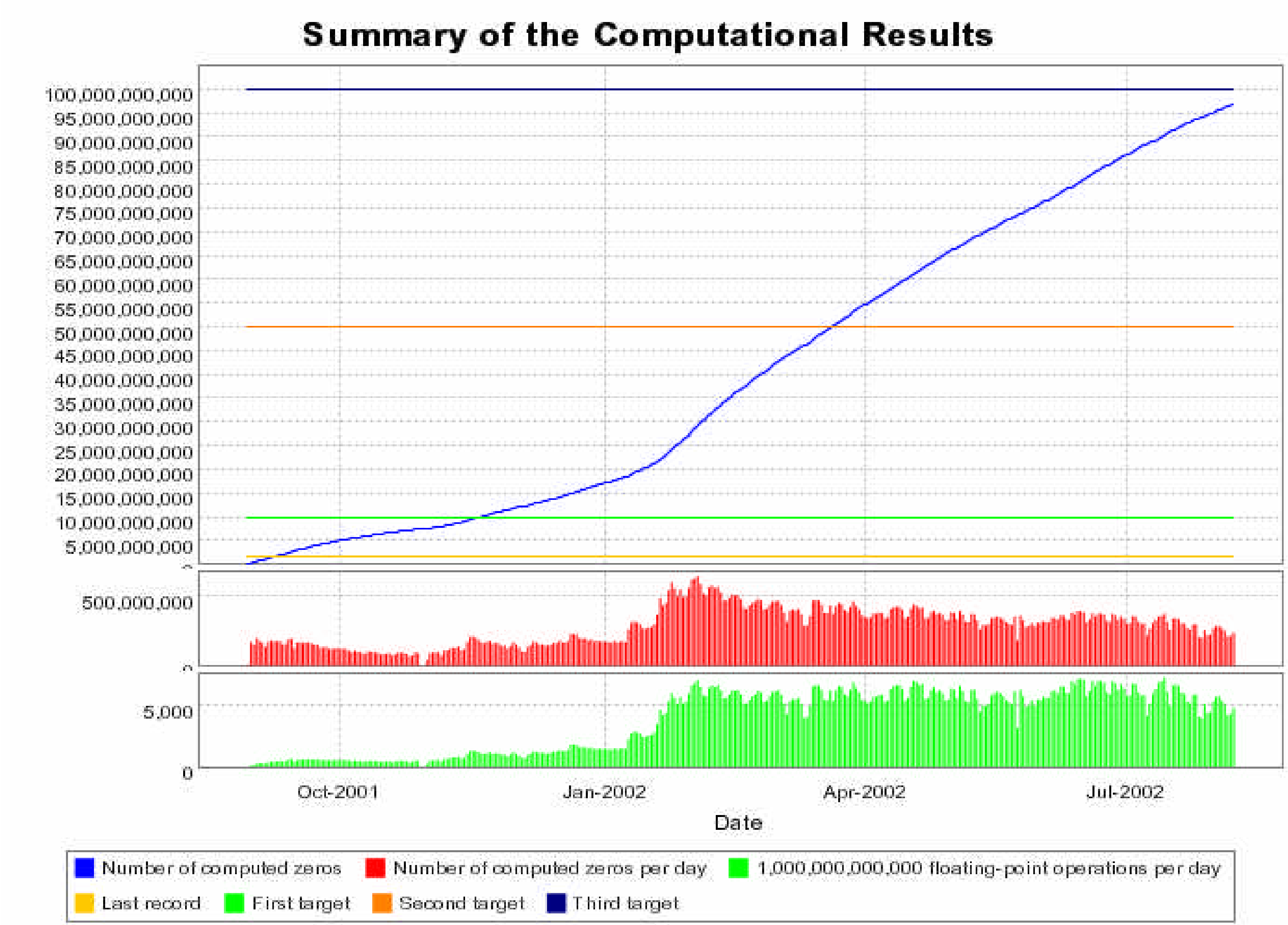
- **Participating in ZetaGrid (8/6/2002):**
  - 288 users and 679 computers

- **1.3×10^{18} floating-point operations (8/6/2002) for calculating more than 97 billion zeros of the Riemann zeta function in 342 days**
  - ~44 GFLOPS
  - ~50 hours maximal performance of IBM ASCI White, 8192 Power3 375 MHz processors (place 1, 11/2001, www.top500.org)
  - ~165 years maximal performance of one Intel Pentium 4 with 2 GHz processors, 250 MFLOPS

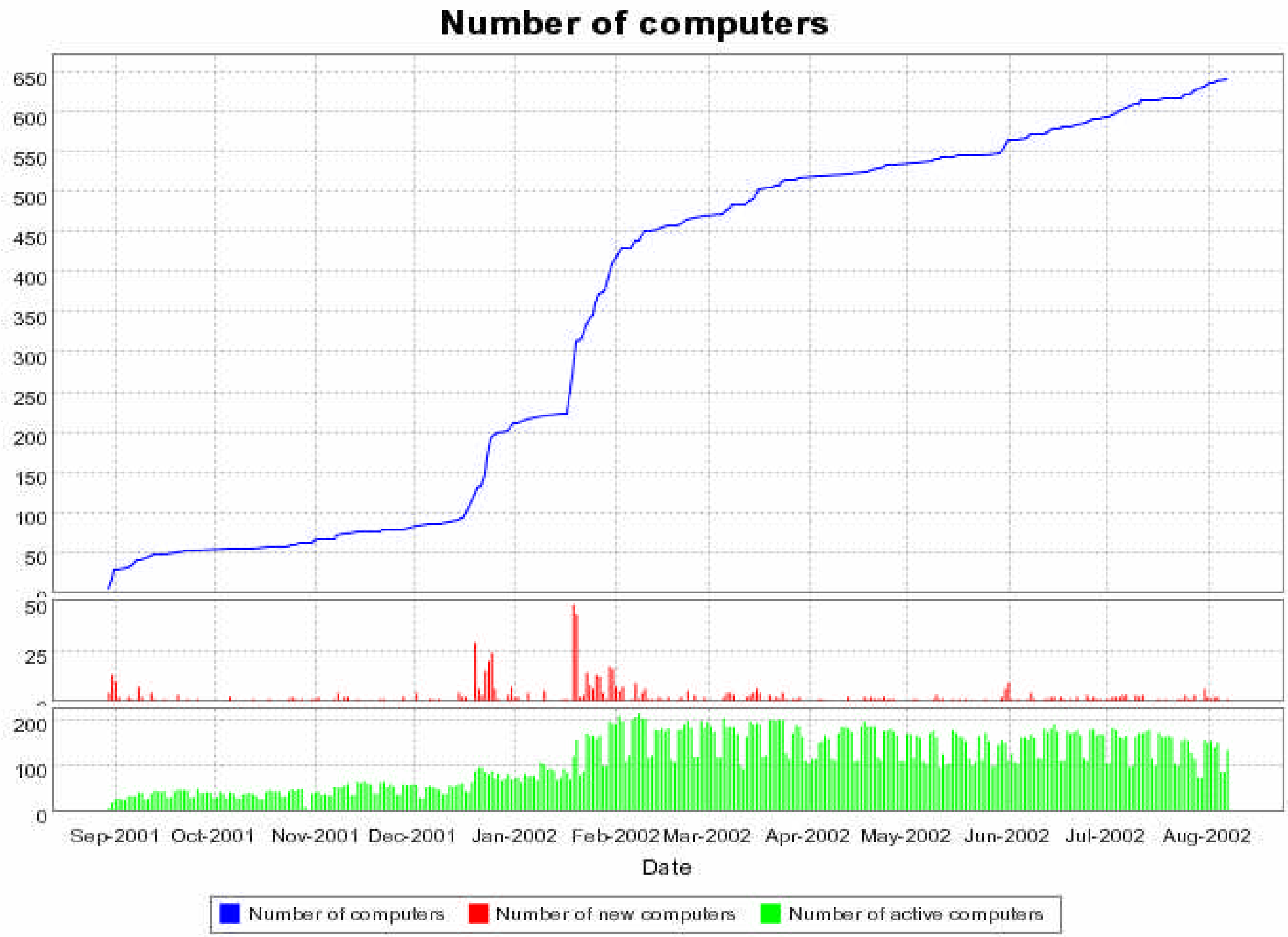
- **Day with best performance (1/31/2002):**
  - 7×10^{15} floating-point operations for calculating more than 642 million zeros
  - ~81 GFLOPS

- **Hour with best performance (5/6/2002, 9:00-9:59 a.m.):**
  - 6.62×10^{14} floating-point operations
  - ~184 GFLOPS (place 184, 11/2001, www.top500.org)

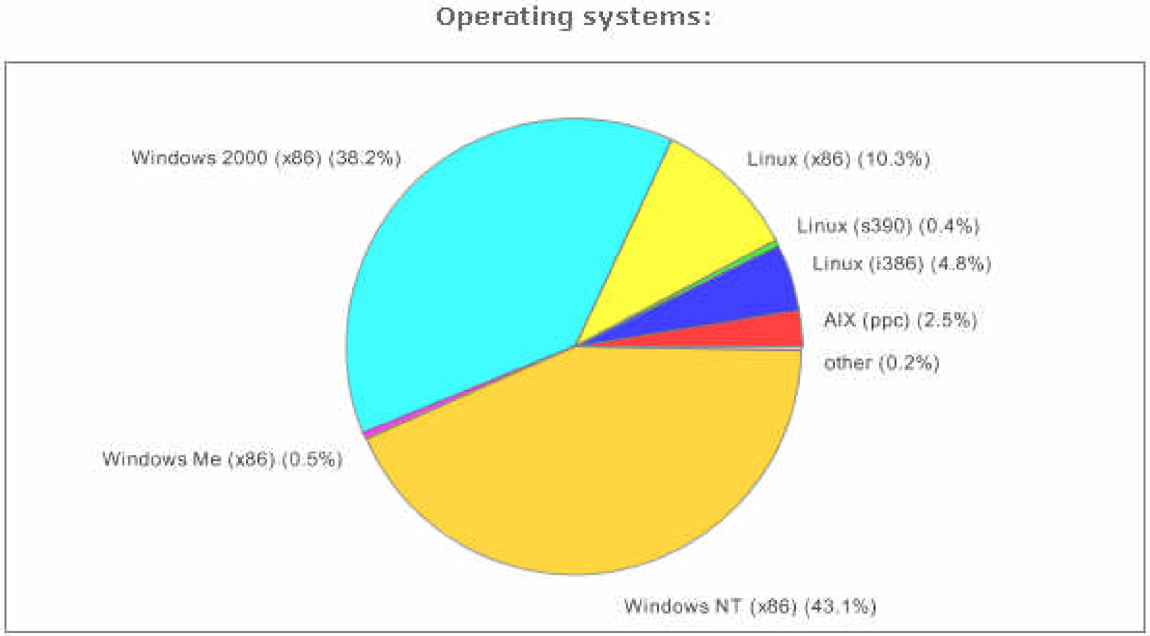
Progress of the Computation



Number of Computers



Supported Computer Architectures



name	processor	computers	range
AIX	ppc	164	2440700000
Linux	i386	31	4694293900
Linux	s390	3	386500000
Linux	x86	56	9978902000
Windows 2000	x86	285	37186321600
Windows 95	x86	4	108500000
Windows 98	x86	4	123300000
Windows Me	x86	1	480000000
Windows NT	x86	105	41949764000
Σ 9		653	97348281500

Questions / Discussion

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